Measuring the performance of industrial processes with data envelopment analysis

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Abstract—Conventional Data Envelopment Analysis (DEA) treats the production system as a black box in measuring efficiency, in that only the inputs supplied to and the outputs produced are considered. By taking the operations of the component processes of the system into account, several network DEA models have been developed. Of these, the Slacks-Based Measure (SBM) approach has attracted much attention for its ability to provide suitable efficiency measures, especially for weakly efficient production units. This paper proposes a general SBM model for network systems, and is able to decompose the system efficiency into a weighted average of the process efficiencies. This relationship holds for all types of network structure. An example of utility companies shows that the network model has stronger discriminating power than the conventional black-box model, and the system efficiency is indeed a weighted average of the process efficiencies. The decomposition of the system efficiency helps identify key factors to improve the performance of a production unit.

Keywords—data envelopment analysis, slacks-based measure, network, efficiency decomposition.

I. INTRODUCTION

Since the seminal work of Charnes et al. [1], data envelopment analysis (DEA) has become the principal technique for measuring the relative efficiency of a set of decision making units (DMUs) that utilize multiple inputs to produce multiple outputs. Originally, the DEA technique was developed to measure the efficiency of a system treated as a black box, without considering its internal structure. When the operations of the component processes are taken into account, one faces a network system, and the conventional model must then be extended to suit the network structure.

Several models for measuring the system and process efficiencies of a network have been developed (see, for example, the review and classification of Castelli et al. [2] and Kao and Hwang [3]). All of these studies are based on radial measures, which have a problem of not being able to appropriately measure the efficiency of weakly efficient DMUs. To solve this problem, Tone [4] proposed a slacks-based measure (SBM) approach to measure the efficiency. In this paper, we develop an SBM model without restrictions on returns to scale, which is able to measure the system and process efficiencies of a general network system at the same time. Moreover, the system efficiency can be decomposed into a weighted average of the process efficiencies. In the next section, an SBM model for general network systems is proposed, and the relationship between the system and process efficiencies is derived.

II. SLACKS-BASED MEASURE OF EFFICIENCY

Denote $X_{ij}$ as the $i$th input, $i=1, \ldots, m$, and $Y_{rj}$ as the $r$th output, $r=1, \ldots, s$, of the $j$th DMU, $j=1, \ldots, n$. The SBM model of Tone [4] is:

\[
E_{k}^{SBM} = \min \frac{1-(1/m)\sum_{i=1}^{m} s_i^+ / X_{ik}}{1+(1/s)\sum_{r=1}^{s} s_r^- / Y_{rk}}
\]

s.t. \[
\begin{align*}
\sum_{j=1}^{n} \lambda_{j} X_{ij} + s_i^- &= X_{ik}, \quad i = 1, \ldots, m \\
\sum_{j=1}^{n} \lambda_{j} Y_{rj} - s_r^+ &= Y_{rk}, \quad r = 1, \ldots, s \\
s_i^+, s_r^+, \lambda_{j} &\geq 0, \quad i = 1, \ldots, m, \quad r = 1, \ldots, s, \quad j = 1, \ldots, n
\end{align*}
\]

Obviously, $E_{k}^{SBM}$ lies in the range of 0 and 1, and DMU $k$ is efficient if $E_{k}^{SBM} = 1$. When the operations of the component processes are taken into consideration in measuring efficiency, the model becomes a little complicated. Figure 1 depicts a general network system, where any process $p$ utilizes exogenous inputs $X_{i(p)}$, $i \in I(p)$, supplied from outside and endogenous inputs $Z_{f(p)}$, $f \in M(p)$, produced by other processes to produce exogenous outputs $Y_{r(p)}$, $r \in O(p)$, as final outputs of the system and endogenous outputs $Z_{g(p)}$, $g \in N(p)$, to be utilized by other processes. Note that in the very general case, $Z_{f(p)}$ can be supplied from different processes, denoted as $Z_{f(p)}^{(1)}$.

![Fig 1. A general network system](image-url)
Let $l^{(p)}_i$, $g^{(p)}_i$, $m^{(p)}_i$, and $n^{(p)}_i$ denote the number of indices in $I^{(p)}$, $O^{(p)}$, $M^{(p)}$, and $N^{(p)}$, respectively, and $s^{(p)}_i$, $t^{(p)}_i$, $s^{(p)}_r$, and $s^{(p)}_g$ be the slack variables. The model for calculating the system efficiency of DMU $k$ can be formulated as:

$$E_k^S = \min \frac{\sum_{p=1}^{q} [1 - \left( \sum_{i} \frac{y_{i}}{x_{i}^{(p)}} + \sum_{j} \frac{z_{j}}{y_{j}^{(p)}} \right) (l^{(p)} + m^{(p)})]}{\sum_{p=1}^{q} [1 + \left( \sum_{i} \frac{y_{i}}{x_{i}^{(p)}} + \sum_{g} \frac{z_{g}}{x_{g}^{(p)}} \right) (n^{(p)} + \hat{n}^{(p)})]}$$

s.t.

$$\sum_{i} \lambda_{i}^{(p)} x_{i}^{(p)} + s_{i}^{(p)} = x_{i}^{(p)}$$, $i \in I^{(p)}$, $p = 1, \ldots, q$

$$\sum_{j} \lambda_{j}^{(p)} z_{j}^{(p)} - t_{j}^{(p)} = z_{j}^{(p)}$$, $j \in M^{(p)}$, $p = 1, \ldots, q$

$$\sum_{g} \lambda_{g}^{(p)} z_{g}^{(p)} - g_{g}^{(p)} = g_{g}^{(p)}$$, $g \in N^{(p)}$, $p = 1, \ldots, q$

$$\sum_{r} \lambda_{r}^{(p)} y_{r}^{(p)} - s_{r}^{(p)} = y_{r}^{(p)}$$, $r \in O^{(p)}$, $p = 1, \ldots, q$

$$\lambda_{j}^{(p)} \geq 0$$, $j = 1, \ldots, n$, $p = 1, \ldots, q$ \hfill (2)

After the optimal solution $(\lambda^{(p)}_{j})^{*}$, $(s^{(p)}_{i})^{*}$, $(t^{(p)}_{j})^{*}$, $(n^{(p)}_{i})^{*}$, $(s^{(p)}_{r})^{*}$, and $(s^{(p)}_{g})^{*}$ is obtained, the system efficiency, $E_{k}^{S}$, and process $p$ efficiency, $E_{k}^{P}$, are calculated as:

$$E_{k}^{S} = \frac{\sum_{p=1}^{q} [1 - \left( \sum_{i} \frac{y_{i}}{x_{i}^{(p)}} + \sum_{j} \frac{z_{j}}{y_{j}^{(p)}} \right) (l^{(p)} + m^{(p)})]}{\sum_{p=1}^{q} [1 + \left( \sum_{i} \frac{y_{i}}{x_{i}^{(p)}} + \sum_{g} \frac{z_{g}}{x_{g}^{(p)}} \right) (n^{(p)} + \hat{n}^{(p)})]}$$ \hfill (3a)

$$E_{k}^{P} = \frac{1 - \left( \sum_{i} \frac{y_{i}}{x_{i}^{(p)}} + \sum_{j} \frac{z_{j}}{y_{j}^{(p)}} \right) (l^{(p)} + m^{(p)})}{1 + \left( \sum_{i} \frac{y_{i}}{x_{i}^{(p)}} + \sum_{g} \frac{z_{g}}{x_{g}^{(p)}} \right) (n^{(p)} + \hat{n}^{(p)})}$$ \hfill (3b)

If we define the weight for process $p$ as:

$$\omega_{p} = \frac{1 + \left( \sum_{r} \frac{y_{r}}{x_{r}^{(p)}} + \sum_{g} \frac{z_{g}}{x_{g}^{(p)}} \right) (n^{(p)} + \hat{n}^{(p)})}{\sum_{p=1}^{q} \left[ 1 + \left( \sum_{r} \frac{y_{r}}{x_{r}^{(p)}} + \sum_{g} \frac{z_{g}}{x_{g}^{(p)}} \right) (n^{(p)} + \hat{n}^{(p)}) \right]}$$ \hfill (4)

then we have $\omega_{p}^{(p)} \geq 0$ and $\sum_{p=1}^{q} \omega_{p}^{(p)} = 1$. In this case, the average of $E_{k}^{P}$ weighted by $\omega_{p}^{(p)}$ is: $\sum_{p=1}^{q} \omega_{p}^{(p)} E_{k}^{P} = E_{k}^{S}$.

That is, the system efficiency of a network system is a weighted average of its component process efficiencies. This relationship holds for all network systems, regardless of their structure.
By applying Equation (3b), the efficiencies of the three processes can be calculated, and the results of the ten companies are shown in the last three columns of Table II. Figure 3 is a pictorial presentation of the system (in bold) and three processes efficiencies of the ten companies for easy comparison. The number in parentheses below each process efficiency is the weight associated with that process calculated from Equation (4). As expected, the system efficiency is a weighted average of the three process efficiencies. Taking Company 1 as an example, one has

$$0.6537 = 0.6328 \times 0.3124 + 0.3686 \times 0.491 + 0.5647 \times 0.253.$$  

The system and process efficiencies in Table II are different from those reported in Tone and Tsutsui [5]. There are two reasons for this. One is that the model used in Tone and Tsutsui is input-oriented; in other words, the slacks associated with the outputs are ignored. The second is that the model of Tone and Tsutsui assumes the slacks associated with the intermediate products to be zero. For these two reasons, the efficiencies of the two studies are not comparable; however those of this study are, in general, smaller than those of Tone and Tsutsui.

IV. CONCLUSION

Network DEA models are able to explain why some DMUs with perfect efficiency scores calculated from the conventional black-box models are actually inefficient by taking the operations of the component processes of the network into consideration in calculating efficiency. The results are more realistic in reflecting the performance of a system.

This paper investigates a slacks-based network DEA model, and derives a property that the system efficiency is a weighted average of the process efficiencies. The process efficiencies and their associated weights help identify the most influential factors in the performance of the system, and a good control of these factors will effectively improve the performance of the system.

REFERENCES


