An analysis of MUSIC, ESPRIT and root-MUSIC Direction of Arrival estimation techniques in Smart Antennas

Mr. Robert Macharia Maina and Dr. Kibet Lang'at and Dr. P. K. Kihato

Abstract—Smart Antennas are a core part of modern wireless communication systems owing to the ever increasing demands on network capacity. Smart Antennas are essentially spatial filters encompassing antenna array structures and beamforming techniques aimed at optimizing radiation/ reception in a wireless communication link. A variety of beamforming techniques rely on knowledge of the Directions of Arrival (DoAs) of desired and interfering signals. Methods that have been proposed in literature to keep track of mobile DoAs include MUltiple SIgnal Classification (MUSIC), Estimation of Signal Parameter via Rotational Invariance Technique (ESPRIT) and root-MUSIC. This paper essentially seeks to analyze the performance of the three listed DoA estimation methods from the point of view of varying: Signal to Noise Ratio (SNR), antenna array size and DoA angular separation. MATLAB software environment is used as the main analysis tool.

Keywords—Direction of Arrival, ESPRIT, MUSIC, root-MUSIC, Smart Antenna

I. INTRODUCTION

Smart Antennas (essentially Spatial Division Multiple Access (SDMA) aids) have been adopted as a worthy tool towards increasing wireless communication networks capacity. Among receive beamforming techniques utilized in smart antennas are those that require signal Direction of Arrival (DoA) information such as the null steering technique. DoA information is usually unavailable particularly in mobile communication networks. There are techniques that have been developed to aid in DoA estimation. Such methods include: Bartlett, Capon (Minimum Variance Distortion-less Response (MVDR)), Min-norm, MUltiple SIgnal Classification (MUSIC), root-MUSIC and Estimation of Signal Parameters via Rotational Invariance Techniques (ESPRIT) [1].

MUSIC, root-MUSIC and ESPRIT DoA estimation methods have been widely adopted and are the focus of this paper. Reviews pertaining the three DoA estimation methods are given in the next three Sections followed by pertinent evaluations in situations involving varying: Signal to Noise Ratio (SNR), antenna array size and DoA angular separation.

Fig. 1 depicts a typical DoA estimation problem featuring a 4-element linear antenna array and three signal DoAs that ought to be estimated.

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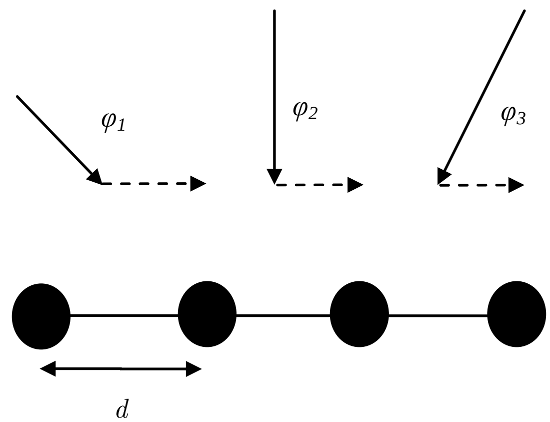

Fig. 1. DoA estimation problem using a linear antenna array

II. MUSIC DoA ESTIMATION METHOD

Considering a scenario involving M noise corrupted signals incident on a N element linear antenna array, the resultant mathematical model is as per (1) or (2). The components of (2) can be expressed as per (3) and (4).

$$\mathbf{X} = \sum_{m=1}^{M} \alpha_m \mathbf{s}(\phi_m) + \mathbf{n} \tag{1}$$

$$\mathbf{X} = \mathbf{S}\alpha + \mathbf{n} \tag{2}$$

$$\mathbf{S} = [\mathbf{s}(\phi_1), \mathbf{s}(\phi_2)...\mathbf{s}(\phi_m)] \tag{3}$$

$$\alpha = [\alpha_1, \alpha_2...\alpha_m]^T \tag{4}$$

$\mathbf{S}$ is an N by M matrix encompassing some M steering vectors.

The correlation matrix corresponding to $\mathbf{X}$ is as per (5).

$$\mathbf{R} = E[\mathbf{X}\mathbf{X}^H] \tag{5}$$

Substituting $\mathbf{X}$ in (5) with (2) yields (6).

$$\mathbf{R} = E[\mathbf{X}\mathbf{X}^H] = E[\mathbf{S}\alpha\alpha^H\mathbf{S}^H] + E[\mathbf{n}\mathbf{n}^H] \tag{6}$$

In a summarized form, (6) can be framed as (7).

$$\mathbf{R} = \mathbf{S}\mathbf{A}\mathbf{S}^H + \sigma^2\mathbf{I} = \mathbf{R}_s + \sigma^2\mathbf{I} \tag{7}$$

The matrix $\mathbf{A}$ in (7) is equivalent to (8).

$$\mathbf{A} = \begin{bmatrix} E[\alpha_1^2] & 0 & ... & 0 \\ 0 & E[\alpha_2^2] & ... & 0 \\ & & . & \\ & & . & \\ 0 & 0 & ... & E[\alpha_M^2] \end{bmatrix} \tag{8}$$

$\mathbf{R}_s$ (signal covariance matrix) is an N by N matrix of rank M. Associated with $\mathbf{R}_s$ are N-M eigenvectors ($\mathbf{q}_m$) corresponding to the zero eigenvalue (9).

$$\mathbf{R}_s\mathbf{q}_m = \mathbf{SAS}^H\mathbf{q}_m = 0 \tag{9}$$

Consequently, $\mathbf{q}_m^H\mathbf{SAS}^H\mathbf{q}_m = 0$ and $\mathbf{S}^H\mathbf{q}_m = 0$ since $\mathbf{A}$ is positive definite.

The implication of $\mathbf{S}^H\mathbf{q}_m = 0$ is that all the N-M eigenvectors ($\mathbf{q}_m$) corresponding to the zero eigenvalue are orthogonal to all M signal steering vectors, the foundation of MUSIC. The pseudo-spectrum corresponding to MUSIC is as per (10 and 11).

$$P_{MUSIC}(\phi) = \frac{1}{\Sigma_{m=1}^{N-M}|\mathbf{s}^H(\phi)\mathbf{q}_m|^2} \tag{10}$$

$$P_{MUSIC}(\phi) = \frac{1}{\mathbf{s}^H(\phi)\mathbf{Q}_n\mathbf{Q}_n^H\mathbf{s}(\phi)} \tag{11}$$

In (11), $\mathbf{Q}_n$, is the matrix of the eigenvectors.

Owing to the fact that the eigenvectors making up the matrix $\mathbf{Q}_n$ are orthogonal to the signal steering vectors, the denominator corresponding to (11) is zero when ϕ is a signal direction. Consequently, estimated signal directions correspond to the highest peaks contained in the pseudo-spectrum. The matrix $\mathbf{R}_s$ is usually estimated from a convenient number of snapshots of received data (12).

$$\mathbf{R} = \frac{1}{K}\Sigma_{k=1}^{K}\mathbf{X}_k\mathbf{X}_k^H \tag{12}$$

Studies utilizing MUSIC DoA estimation method can be found in [2], [3], [4], [5] and [6].

III. ROOT-MUSIC DoA ESTIMATION METHOD

Shortcomings associated with the MUSIC DoA estimation method are the fact that accuracy is limited by the extent of discretization at which (11) is evaluated and a search algorithm (or human intervention) is a necessity in identifying the peaks of (11). Root-MUSIC gives numeric values corresponding to estimated DoAs.

Defining some parameter z as per (13), and with reference to Fig. 1, a steering vector can be defined as per (14) given that the array in question is uniform and with isotropic elements.

$$z = e^{jkdcos\phi} \tag{13}$$

$$\mathbf{s}(\phi) = \begin{bmatrix} 1 & z & z^2 & . & . & . & z^{N-1} \end{bmatrix}^T \tag{14}$$

Consequently,

$$\mathbf{q}_m^H\mathbf{s} = \sum_{n=0}^{N-1} q_{mn}^* z^n = q_m(z) \tag{15}$$

The product $\mathbf{q}_m^H\mathbf{s}$ is equivalent to a polynomial in z as per (15). The directions (ϕ) where $\mathbf{q}_m \perp \mathbf{s}(\phi)$ are desired, essentially this translates to looking for the roots of a polynomial going by (15).

The polynomial whose roots are desired is arrived at in (16) to (18).

$$P_{MUSIC}^{-1}(\phi) = \mathbf{s}^H(\phi)\mathbf{Q}_n\mathbf{Q}_n^H\mathbf{s}(\phi) = \mathbf{s}^H(\phi)\mathbf{Cs}(\phi) \tag{16}$$

$$P_{MUSIC}^{-1}(\phi) = \sum_{m=0}^{N-1}\sum_{n=0}^{N-1} z^n C_{mn} z^{-m} \tag{17}$$

$$P_{MUSIC}^{-1}(\phi) = \sum_{m=0}^{N-1}\sum_{n=0}^{N-1} z^{n-m} C_{mn} \tag{18}$$

Setting setting $l = n - m$ yields (19) (a polynomial of degree $(2N-2)$ with $(2N-2)$ zeros z).

$$P_{MUSIC}^{-1}(\phi) = \sum_{l=-(N-1)}^{N+1} z^l C_l \tag{19}$$

The DoAs are obtained as per (20).

$$\phi_m = cos^{-1}[\frac{\Im\ln(z_m)}{kd}] \quad m = 1, ...M \tag{20}$$

Studies utilizing root-MUSIC DoA estimation method can be found in [7] and [8].

IV. ESPRIT DoA ESTIMATION METHOD

ESPRIT DoA estimation method is grounded on the fact that the steering vector features elements with progressive uniform phase shift (14). The matrix of steering vectors can be presented as per (21).

$$\mathbf{S} = \begin{bmatrix} 1 & 1 & ... & 1 \\ z_1 & z_2 & ... & z_M \\ & & . & \\ & & . & \\ z_1^{N-1} & z_2^{N-1} & ... & z_M^{N-1} \end{bmatrix} \tag{21}$$

Based on $\mathbf{S}$ in (21), we define $\mathbf{S}_0$ and $\mathbf{S}_1$ as per (22) and (23).

$$\mathbf{S}_0 = \begin{bmatrix} 1 & 1 & ... & 1 \\ z_1 & z_2 & ... & z_M \\ & & . & \\ & & . & \\ z_1^{N-2} & z_2^{N-2} & ... & z_M^{N-2} \end{bmatrix} \tag{22}$$

$$\mathbf{S}_1 = \begin{bmatrix} z_1 & z_2 & ... & z_M \\ & & . & \\ & & . & \\ & & . & \\ z_1^{N-1} & z_2^{N-1} & ... & z_M^{N-1} \end{bmatrix} \tag{23}$$

It can be noted that $\mathbf{S}_1 = \mathbf{S}_0\Phi$, where Φ is as per (24).

$$\Phi = \begin{bmatrix} z_1 & 0 & ... & 0 \\ 0 & z_2 & ... & 0 \\ & & . & \\ & & . & \\ 0 & 0 & ... & z_M \end{bmatrix} \tag{24}$$

The diagonal entries of Φ correspond to element to element phase shift. If Φ can be estimated, DoAs can be estimated on the basis of (13). But $\mathbf{S}_0$ and $\mathbf{S}_1$ are unknown. The approach towards realizing an ESPRIT solution flows from (25).

$$\mathbf{Q}_s = \mathbf{SC} \tag{25}$$

Where $\mathbf{Q}_s$ is a matrix of signal eigenvectors, and $\mathbf{C}$ is some invertible matrix.

Consequently,

$$\mathbf{Q}_0 = \mathbf{S}_0\mathbf{C} \tag{26}$$

$$\mathbf{Q}_1 = \mathbf{S}_1\mathbf{C} = \mathbf{S}_0\Phi\mathbf{C} \tag{27}$$

Considering (28),

$$\mathbf{Q}_1\mathbf{C}^{-1}\Phi^{-1}\mathbf{C} = \mathbf{S}_0\Phi\mathbf{C}\mathbf{C}^{-1}\Phi^{-1}\mathbf{C} = \mathbf{S}_0\mathbf{C} = \mathbf{Q}_0 \tag{28}$$

Letting

$$\Psi^{-1} = \mathbf{C}^{-1}\Phi^{-1}\mathbf{C} \tag{29}$$

We have

$$\mathbf{Q}_1\Psi^{-1} = \mathbf{Q}_0 \tag{30}$$

and

$$\mathbf{Q}_1 = \mathbf{Q}_0\Psi \tag{31}$$

where

$$\Psi = \mathbf{C}^{-1}\Phi\mathbf{C} \tag{32}$$

Going by (31), Φ is a diagonal matrix of the eigenvalues of Ψ.

The DoAs are obtained as per (20) upon finding the eigenvalues of Ψ.

Studies utilizing ESPRIT DoA estimation method can be found in [9], [10] and [11].

V. METHODOLOGY

This work is modeled on the basis of estimating the DoAs associated with two uncorrelated, equal amplitude signal sources. A uniform linear antenna array framework is utilized (with isotropic elements).

The performance of MUSIC, root-MUSIC and ESPRIT DoA estimation methods is compared graphically via the respective pseudo-spectrum results in a variety of conditions. Initially, SNR is varied in an environment featuring a 15-element linear antenna array. Other cases analyzed involve varying antenna array size and DoA angular separation.

VI. RESULTS

A. SNR variation

The figures presented in this section (Fig. 2 to Fig. 10) correspond to DoA estimation results obtained in varying SNR conditions. A 15 size uniform linear antenna array is assumed, with half a wavelength spacing distance. The two expected DoAs are -10 and +10 degrees.

Fig. 2 to Fig. 4 correspond to DoA estimation results obtained in zero noise conditions.

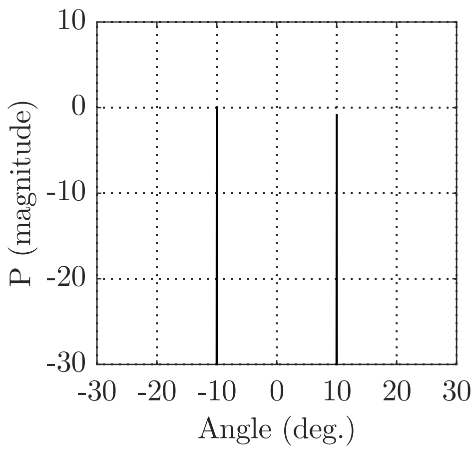

Fig. 2. Zero noise MUSIC result.

The two expected DoAs (-10 and +10 degrees) can be easily deciphered from the MUSIC result in Fig. 2.

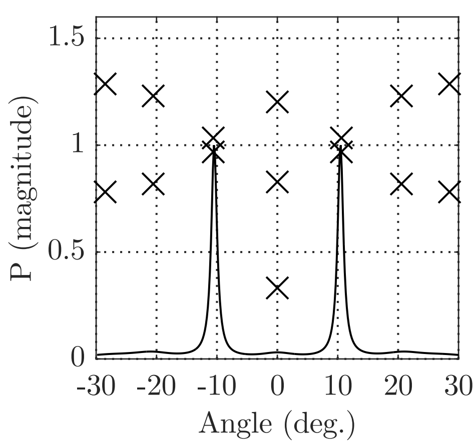

Fig. 3. Zero noise root-MUSIC result.

The two expected DoAs (-10 and +10 degrees) can be easily deciphered from the root-MUSIC result in Fig. 3.

The two expected DoAs (-10 and +10 degrees) can be easily deciphered from the ESPRIT result in Fig. 4.

A comparative observation of the pseudo-spectrums in Fig. 2, Fig. 3 and Fig. 4 (zero noise condition) depict the MUSIC result as the best even though roughly similar results can be obtained graphically from the other two DoA estimation methods.

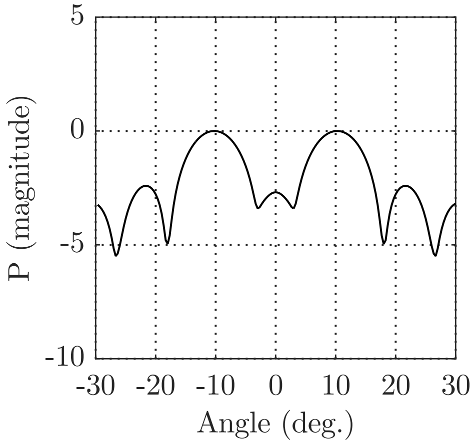

Fig. 4. Zero noise ESPRIT result.

Fig. 5 to Fig. 7 correspond to DoA estimation results obtained in average noise conditions.

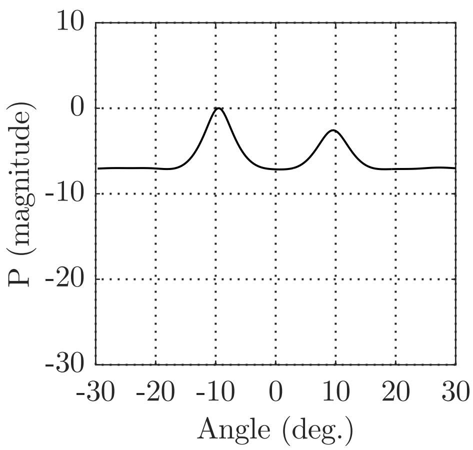

Fig. 5. Average noise MUSIC result.

The two expected DoAs (-10 and +10 degrees) can be somewhat deciphered from the MUSIC result in Fig. 5.

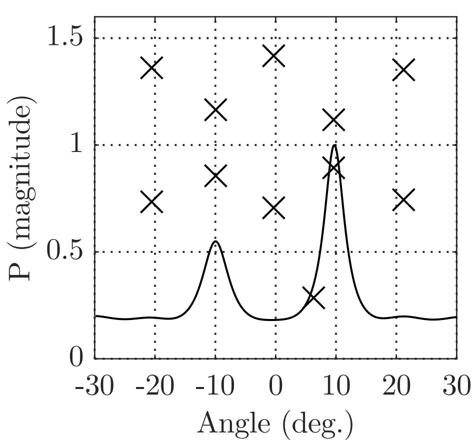

Fig. 6. Average noise root-MUSIC result.

The two expected DoAs (-10 and +10 degrees) can be somewhat deciphered from the root-MUSIC result in Fig. 6.

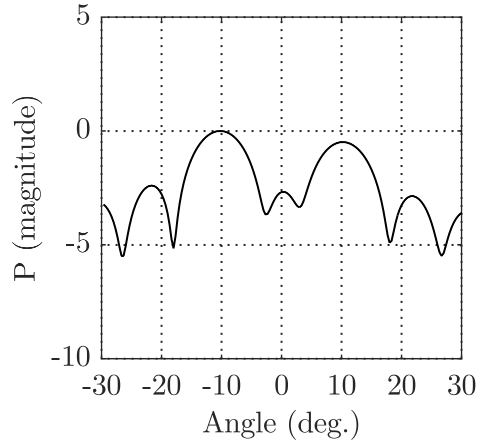

Fig. 7. Average noise ESPRIT result.

The two expected DoAs (-10 and +10 degrees) can be somewhat deciphered from the ESPRIT result in Fig. 7.

A comparative observation of the pseudo-spectrums in Fig. 5, Fig. 6 and Fig. 7 (average noise condition) depict the MUSIC and root-MUSIC results as the best even though roughly similar results can be obtained graphically from the ESPRIT estimation method.

Fig. 8 to Fig. 10 correspond to DoA estimation results obtained in high noise conditions.

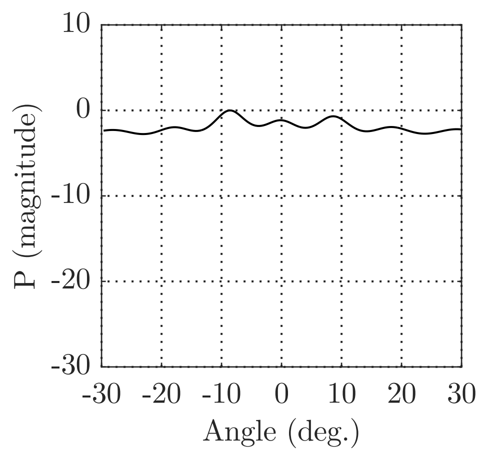

Fig. 8. High noise MUSIC result.

The two expected DoAs (-10 and +10 degrees) are difficult to decipher from the MUSIC result in Fig. 8.

The two expected DoAs (-10 and +10 degrees) can be somewhat deciphered with difficulty from the root-MUSIC result in Fig. 9.

The two expected DoAs (-10 and +10 degrees) can be somewhat deciphered from the ESPRIT result in Fig. 7.

A comparative observation of the pseudo-spectrums in Fig. 8, Fig. 9 and Fig. 10 (high noise condition) depict the ESPRIT result as the best. A roughly good result is obtained with the root-MUSIC method and a poor result with the MUSIC estimation method.

Going by the results obtained with the 3 different methods under various noise conditions, the MUSIC/ root-MUSIC

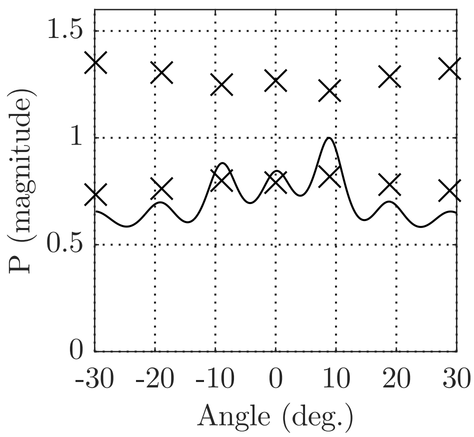

Fig. 9. High noise root-MUSIC result.

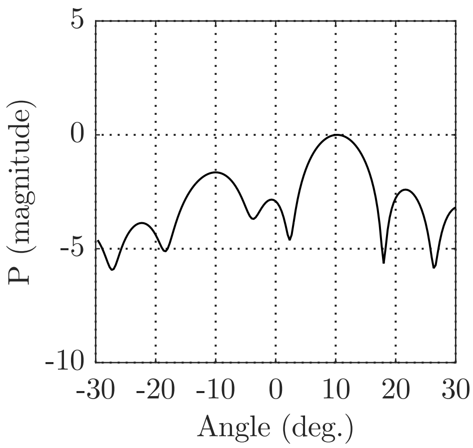

Fig. 10. High noise ESPRIT result.

methods are identified as better options in low noise conditions whereas the ESPRIT method is better suited for application in high noise environments.

B. Array size variation

The figures presented in this section (Fig. 11 to Fig. 19) correspond to DoA estimation results obtained in varying array size scenarios.Half a wavelength array element spacing distance is used. The two expected DoAs are -10 and +10 degrees. Average noise conditions are assumed.

1) 5 element array: A comparative observation of the pseudo-spectrums in Fig. 11, Fig. 12 and Fig. 13 (5 element arrays) depict wide pseudo-spectrum lobe widths. The MUSIC result is somewhat the best even though roughly similar results can be obtained graphically from the root-MUSIC method. The wide pseudo-spectrum lobe width in the ESPRIT estimation method result implies poor DoA estimation accuracy.

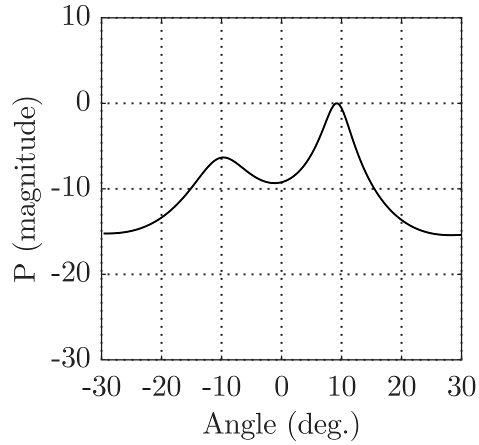

Fig. 11. 5 element array MUSIC result.

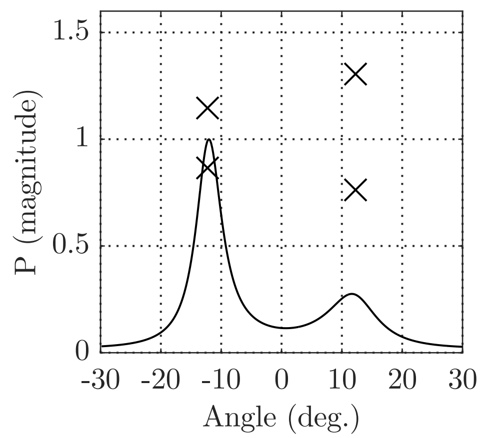

Fig. 12. 5 element array root-MUSIC result.

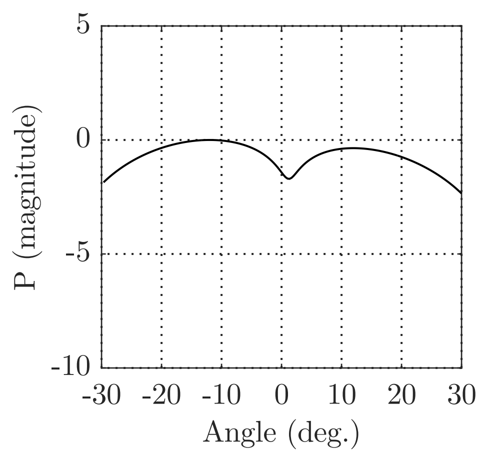

Fig. 13. 5 element array ESPRIT result.

2) 10 element array: A comparative observation of the pseudo-spectrums in Fig. 14, Fig. 15 and Fig. 16 (10 element arrays) depict roughly similar DoA estimation results, with narrower pseudo-spectrum lobe widths than in the 5 element array case.

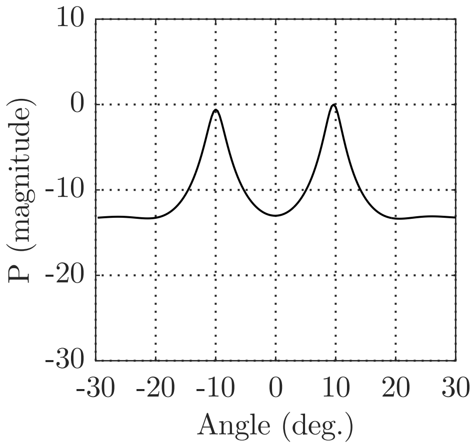

Fig. 14. 10 element array MUSIC result.

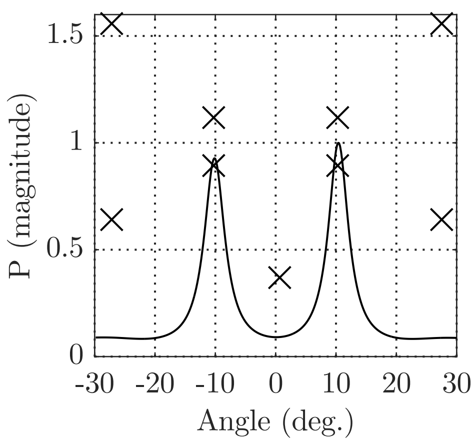

Fig. 15. 10 element array root-MUSIC result.

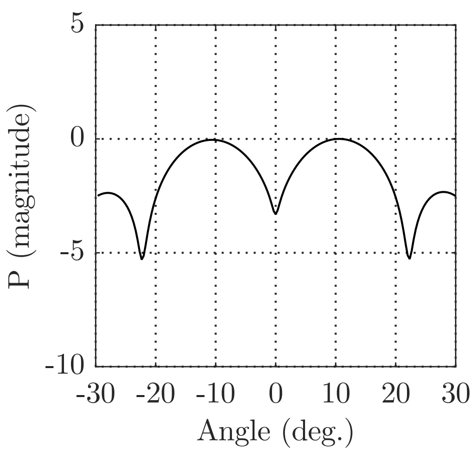

Fig. 16. 10 element array ESPRIT result.

3) 15 element array: A comparative observation of the pseudo-spectrums in Fig. 17, Fig. 18 and Fig. 19 (15 element arrays) depict roughly similar DoA estimation results, with narrower pseudo-spectrum lobe widths than in the 5 and 10 element array cases. It is easily deduced that DoA estimation accuracy improves with increase in antenna array size.

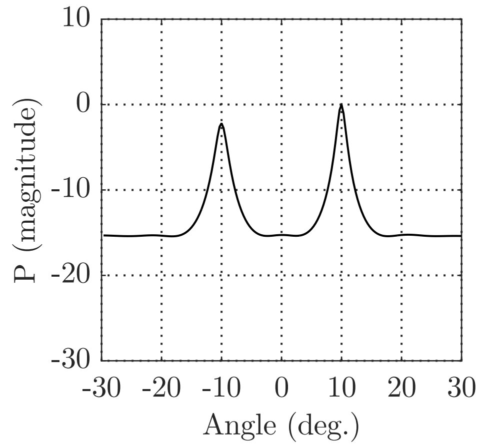

Fig. 17. 15 element array MUSIC result.

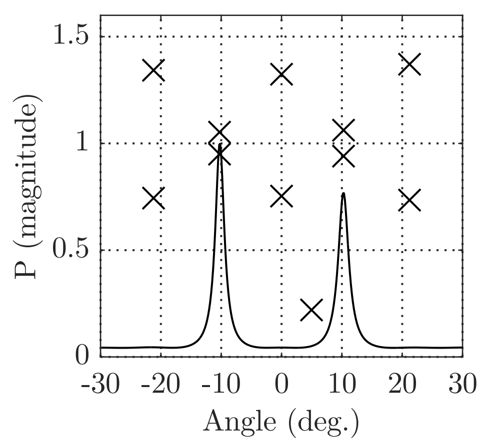

Fig. 18. 15 element array root-MUSIC result.

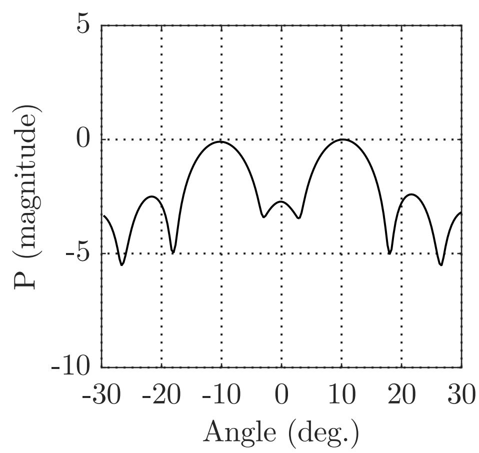

Fig. 19. 15 element array ESPRIT result.

C. DoA angular separation variation

The figures presented in this section (Fig. 20 to Fig. 37) correspond to DoA estimation results obtained in varying DoA angular separation scenarios. A 10 element array with half a wavelength element spacing distance is considered. Zero noise and average noise conditions are considered.

1) 20 degrees separation: Results obtained in a 20 degrees separation scenario (-10 and +10 degrees) in a situation featuring zero noise are depicted in Fig. 20, Fig. 21 and Fig. 22. Good DoA estimation results are obtained in all cases (MUSIC, root-MUSIC and ESPRIT).

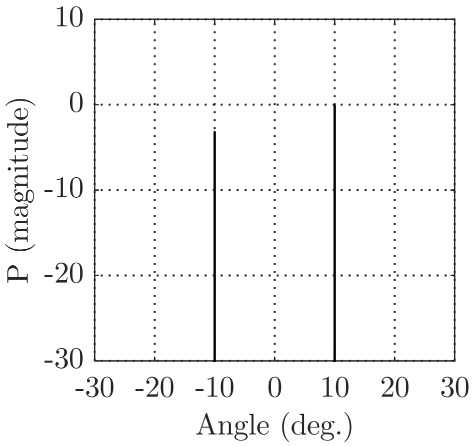

Fig. 20. Zero noise MUSIC result.

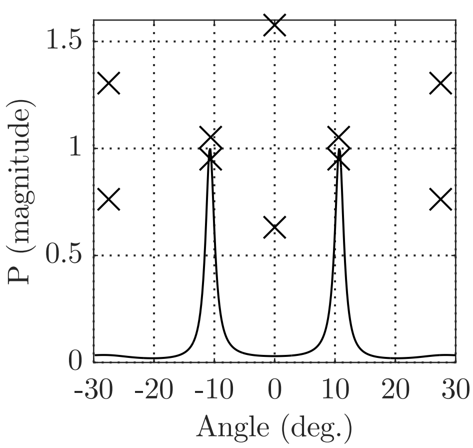

Fig. 21. Zero noise root-MUSIC result.

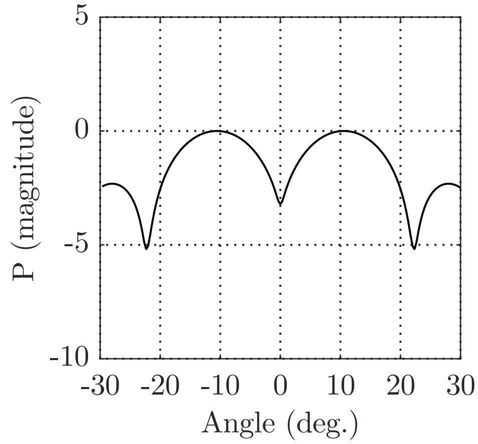

Fig. 22. Zero noise ESPRIT result.

Results obtained in a 20 degrees separation scenario (-10 and +10 degrees) in a situation featuring average noise are depicted in Fig. 23, Fig. 24 and Fig. 25. Fairly good DoA estimation results are obtained in all cases (MUSIC, root-MUSIC and ESPRIT).

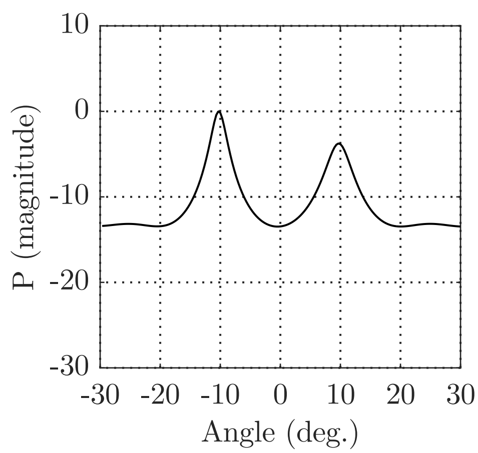

Fig. 23. Average noise MUSIC result.

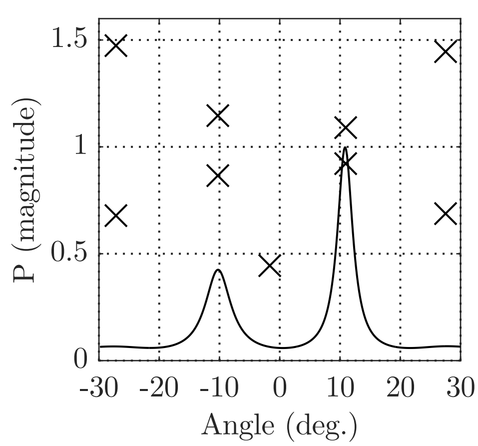

Fig. 24. Average noise root-MUSIC result.

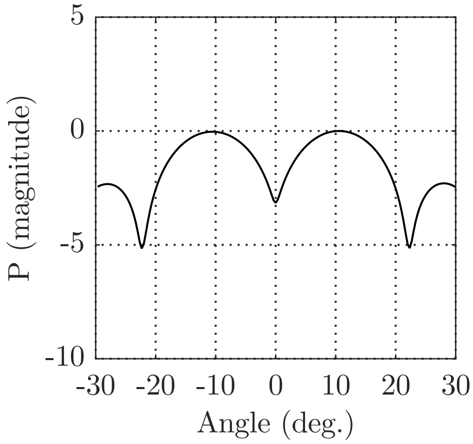

Fig. 25. Average noise ESPRIT result.

2) 10 degrees separation: Results obtained in a 10 degrees separation scenario (-5 and +5 degrees) in a situation featuring zero noise are depicted in Fig. 26, Fig. 27 and Fig. 28. Good DoA estimation results are obtained in MUSIC and root-MUSIC cases with average results being observed in the ESPRIT case.

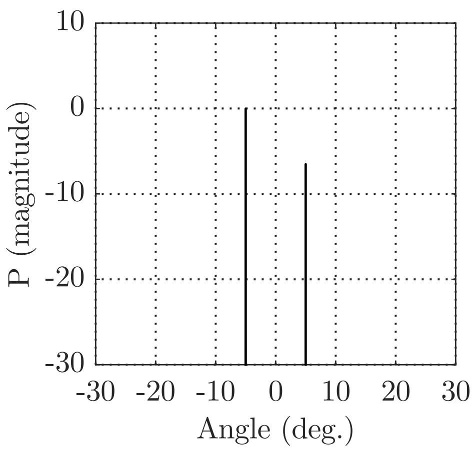

Fig. 26. Zero noise MUSIC result.

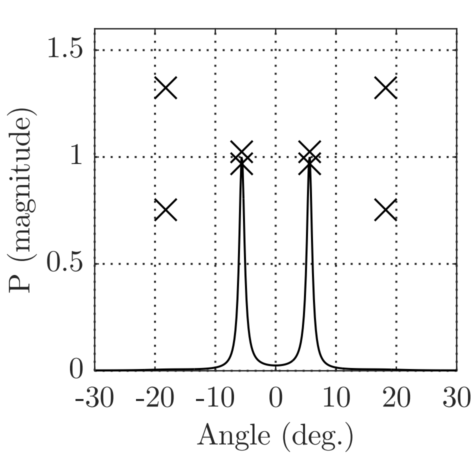

Fig. 27. Zero noise root-MUSIC result.

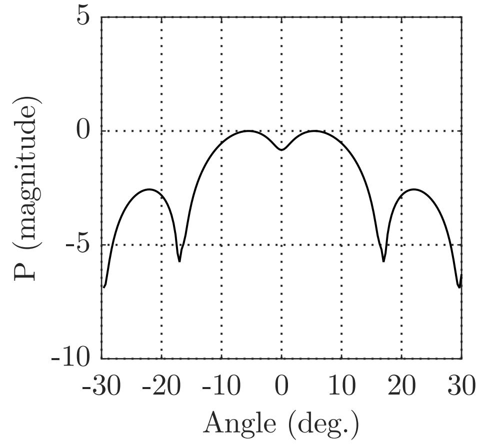

Fig. 28. Zero noise ESPRIT result.

Results obtained in a 10 degrees separation scenario (-5 and +5 degrees) in a situation featuring average noise are depicted in Fig. 29, Fig. 30 and Fig. 31. Fairly good DoA estimation results are obtained in MUSIC and root-MUSIC cases with average results being observed in the ESPRIT case.

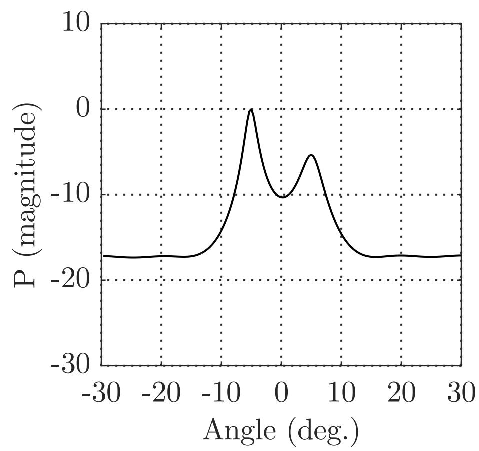

Fig. 29. Average noise MUSIC result.

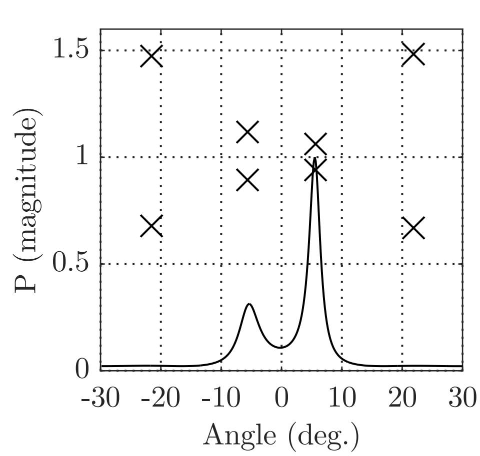

Fig. 30. Average noise root-MUSIC result.

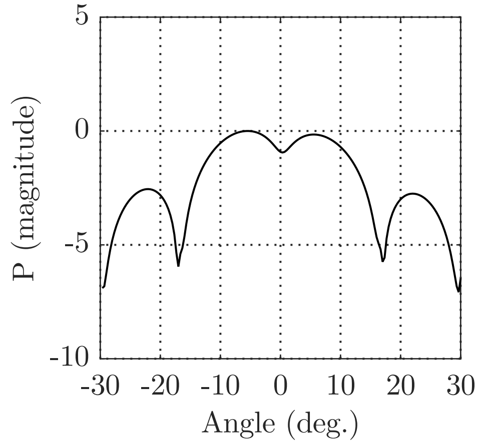

Fig. 31. Average noise ESPRIT result.

3) 4 degrees separation: Results obtained in a 4 degrees separation scenario (-2 and +2 degrees) in a situation featuring zero noise are depicted in Fig. 32, Fig. 33 and Fig. 34. Fairly good DoA estimation results are obtained in MUSIC and root-MUSIC cases with indecipherable results being observed in the ESPRIT case.

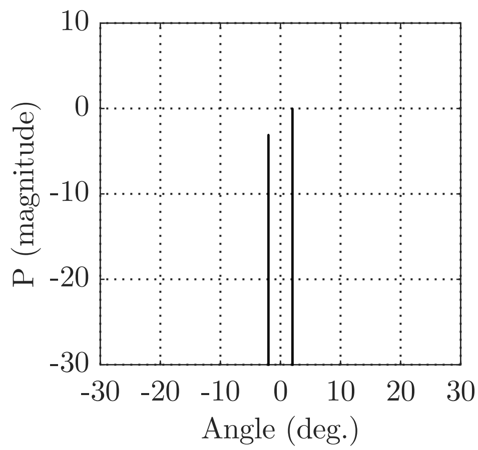

Fig. 32. Zero noise MUSIC result.

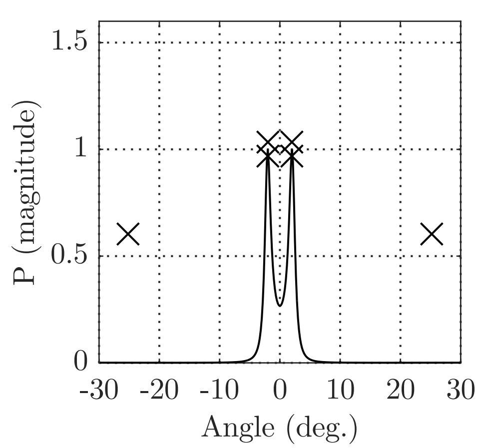

Fig. 33. Zero noise root-MUSIC result.

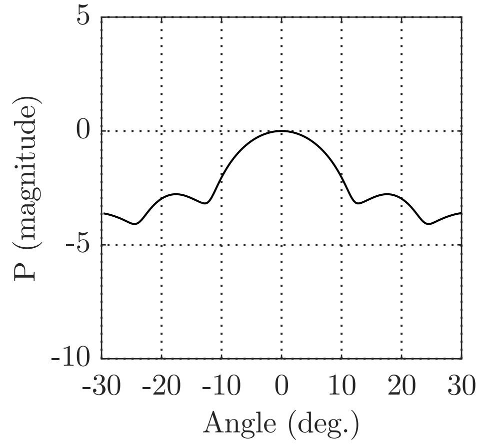

Fig. 34. Zero noise ESPRIT result.

Results obtained in a 4 degrees separation scenario (-2 and +2 degrees) in a situation featuring average noise are depicted in Fig. 35, Fig. 36 and Fig. 37. Indecipherable DoA estimation results are obtained in all cases (MUSIC, root-MUSIC and ESPRIT).

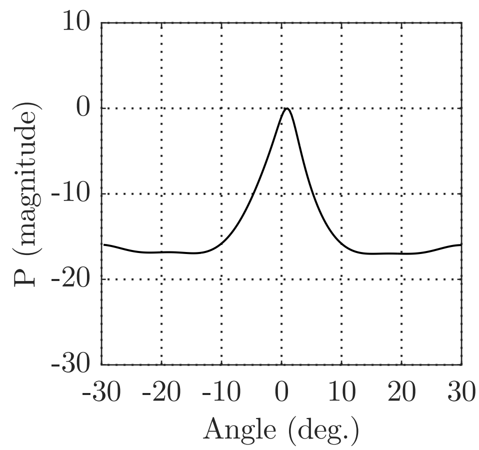

Fig. 35. Average noise MUSIC result.

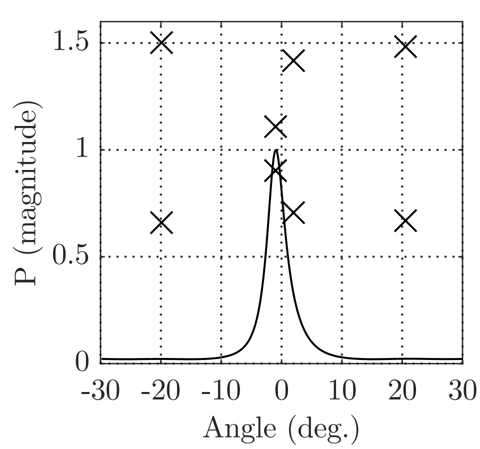

Fig. 36. Average noise root-MUSIC result.

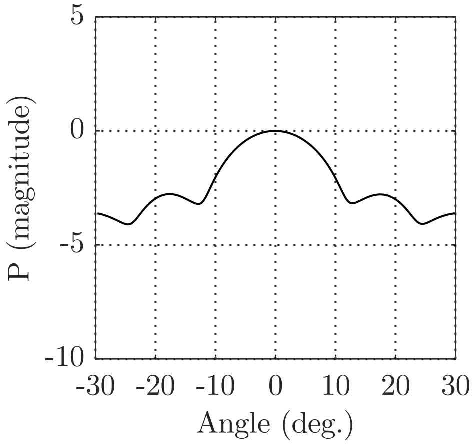

Fig. 37. Average noise ESPRIT result.

VII. CONCLUSION

The performance of MUSIC, root-MUSIC and ESPRIT DoA estimation methods has been looked into under various conditions.

MUSIC/ root-MUSIC methods are identified as better options in low noise (high SNR) conditions whereas the ESPRIT method is better suited for application in high noise (low SNR) environments.

Antenna array size is noted to be of significance: It is deduced that DoA estimation accuracy improves with increase in antenna array size.

MUSIC/ root-MUSIC methods are identified as better options in estimating DoAs associated with closely spaced sources as opposed to the ESPRIT method.

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